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Similarity reductions of the KP equation by a direct method

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Abstract. Basing on a direct method developed by Clarkson and Kruskal, the Kadomtsev–Petviashvili (KP) equation has been reduced to three types of $(1+1)$-dimensional partial differential equations which are equivalent to the three types of the similarity reduction equations obtained by the classical Lie approach but with different independent variables. More arbitrary functions which have been missed by the classical Lie approach have been included in the solutions of the KP equation. For instance, the third type of reduction obtained by the direct method can be divided into three subcases and the third type of solution of the KP equation obtained by the classical Lie approach is only a special case of one subcase of our results.

1. Introduction

The Kadomtsev–Petviashvili equation (KPE) [1]

$$-u_{tx}+6u_x^2+6uu_{xx}+u_{xxxx}+u_{yy}=0 \tag{1}$$

where subscripts denote differentiations, arises in many fields of physics, particularly in fluid mechanics, plasma physics, gas dynamics, etc. The KPE is also of considerable importance in mathematics because it is one of the few equations in more than $(1+1)$ dimensions that is completely integrable. Furthermore, this equation allows an infinite-dimensional Lie group of symmetries. Using the one-dimensional subalgebras of the symmetry algebra, David *et al* [2] reduced the KPE to some partial differential equations (PDEs) in two variables: the Boussinesq equation, a once-differentiated Korteweg–de Vries equation (KdVE) and a linear equation, respectively. We have chosen the form of the KPE as in [3] (for $\alpha^2=\frac{1}{3}$) rather than that of [2], though they are equivalent.

By using the direct method developed by Clarkson and Kruskal [4], we have reduced the KPE to some ordinary differential equations (ODEs) [5]. In order to exploit the connections between the classical Lie approach and the direct method and, to find all the possible symmetry reductions of the KPE, we will reduce the KPE to some $(1+1)$ PDEs by the direct method. All the reductions for reducing the KPE to $(1+1)$-dimensional PDEs are found. All the reduction equations can be transformed to the results obtained by the classical Lie approach but with more general independent variables. In other words, some additional arbitrary functions which are missed by the classical Lie approach can be included in the results obtained by the direct method.

2. Symmetry reductions of the KPE

All the similarity solutions of the form

$$u(x, y, t) = U(x, y, t, w(\xi, \eta)) \qquad \xi = \xi(x, y, t) \qquad \eta = \eta(x, y, t) \tag{2}$$

where U, ξ and η are functions of the indicated variables and $w(\xi, \eta)$ satisfies a PDE in two variables, ξ and η, may be obtained by substituting (2) into (1). However, we can prove that it is sufficient to seek a similarity reduction of the KPE in the special form

$$u(x, y, t) = \alpha(x, y, t) + \beta(x, y, t) w(\xi(x, y, t), \eta(x, y, t)) \tag{3}$$

rather than the most general form (2).

Substituting (3) into (1) yields

$$\begin{aligned}\gamma_0 w_{\xi\xi\xi\xi} + \gamma_1 w_{\eta\eta\eta\eta} + 6\gamma_2(w_\xi^2 + ww_{\xi\xi}) + 6\gamma_3(w_\eta^2 + ww_{\eta\eta}) + 12\gamma_4(w_\xi w_\eta + ww_{\xi\eta}) \\ + 4\gamma_5 w_{\xi\xi\xi\eta} + 4\gamma_6 w_{\eta\eta\eta\xi} + 6\gamma_7 w_{\xi\xi\eta\eta} + 6\gamma_8 w^2 + 6\gamma_9 ww_\xi + 6\gamma_{10} ww_\eta \\ + \gamma_{11} w_{\xi\xi\xi} + \gamma_{12} w_{\eta\eta\eta} + \gamma_{13} w_{\xi\xi\eta} + \gamma_{14} w_{\eta\eta\xi} + \gamma_{15} + \gamma_{16} w + \gamma_{17} w_\xi \\ + \gamma_{18} w_\eta + \gamma_{19} w_{\xi\xi} + \gamma_{20} w_{\eta\eta} + \gamma_{21} w_{\xi\eta} = 0\end{aligned} \tag{4}$$

where

$$\gamma_0 = \beta\xi_x^4 \tag{5}$$

$$\gamma_1 = \beta\eta_x^4 \tag{6}$$

$$\gamma_2 = \beta^2\xi_x^2 \tag{7}$$

$$\gamma_3 = \beta^2\eta_x^2 \tag{8}$$

$$\gamma_4 = \beta^2\eta_x\xi_x \tag{9}$$

$$\gamma_5 = \beta\xi_x^3\eta_x \tag{10}$$

$$\gamma_6 = \beta\eta_x^3\xi_x \tag{11}$$

$$\gamma_7 = \beta\xi_x^2\eta_x^2 \tag{12}$$

$$\gamma_8 = \beta_x^2 + \beta\beta_{xx} \tag{13}$$

$$\gamma_9 = 4\beta\beta_x\xi_x + \beta^2\xi_{xx} \tag{14}$$

$$\gamma_{10} = 4\beta_x\beta\eta_x + \beta^2\eta_{xx} \tag{15}$$

$$\gamma_{11} = 4\beta_x\xi_x^3 + 6\beta\xi_x^2\xi_{xx} \tag{16}$$

$$\gamma_{12} = 4\beta_x\eta_x^3 + 6\beta\eta_x^2\eta_{xx} \tag{17}$$

$$\gamma_{13} = 12\beta\eta_x\xi_x\xi_{xx} + 12\beta_x\eta_x\xi_x^2 + 6\beta\eta_{xx}\xi_x^2 \tag{18}$$

$$\gamma_{14} = 12\beta\xi_x\eta_x\eta_{xx} + 12\beta_x\xi_x\eta_x^2 + 6\beta\xi_{xx}\eta_x^2 \tag{19}$$

$$\gamma_{15} = -\alpha_{tx} + 6\alpha_x^2 + 6\alpha\alpha_{xx} + \alpha_{yy} + \alpha_{xxxx} \tag{20}$$

$$\gamma_{16} = -\beta_{tx} + 12\alpha_x\beta_x + 6\alpha\beta_{xx} + 6\alpha_{xx}\beta + \beta_{yy} + \beta_{xxxx} \tag{21}$$

$$\begin{aligned}\gamma_{17} = -\beta_t\xi_x - \beta_x\xi_t - \beta\xi_{tx} + 12\alpha_x\beta\xi_x + 12\alpha\beta_x\xi_x + 6\alpha\beta\xi_{xx} + 2\beta_y\xi_y + \beta\xi_{yy} \\ + 4\beta_{xxx}\xi_x + 4\beta_x\xi_{xxx} + \beta\xi_{xxxx} + 6\beta_{xx}\xi_{xx}\end{aligned} \tag{22}$$

$$\gamma_{18}=-\beta_t\eta_x-\beta_x\eta_t-\beta\eta_{tx}+12\alpha_x\beta\eta_x+12\alpha\beta_x\eta_x+6\alpha\beta\eta_{xx}+2\beta_y\eta_y+\beta\eta_{yy}$$
$$+4\beta_{xxx}\eta_x+4\beta_x\eta_{xxx}+\beta\eta_{xxxx}+6\beta_{xx}\eta_{xx} \quad (23)$$

$$\gamma_{19}=-\beta\xi_t\xi_x+6\alpha\beta\xi_x^2+4\beta\xi_{xxx}\xi_x+12\beta_x\xi_{xx}\xi_x+6\beta_{xx}\xi_x^2+3\beta\xi_{xx}^2+\beta\xi_y^2 \quad (24)$$

$$\gamma_{20}=-\beta\eta_t\eta_x+6\alpha\beta\eta_x^2+4\beta\eta_{xxx}\eta_x+12\beta_x\eta_{xx}\eta_x+6\beta_{xx}\eta_x^2+3\beta\eta_{xx}^2+\beta\eta_y^2 \quad (25)$$

$$\gamma_{21}=-\beta\xi_t\eta_x-\beta\eta_t\xi_x+12\alpha\beta\xi_x\eta_x+4\beta\eta_{xxx}\xi_x+4\beta\xi_{xxx}\eta_x+12\beta_x\xi_{xx}\eta_x$$
$$+12\beta_x\eta_{xx}\xi_x+6\beta\xi_{xx}\eta_{xx}+12\beta_{xx}\xi_x\eta_x+2\beta\xi_y\eta_y. \quad (26)$$

Equation (4) is a PDE of $w(\xi, \eta)$ in two variables only for the ratios of the coefficients of different partial derivatives and powers of $w(\xi, \eta)$ being functions of ξ and η. If $\xi_x \neq 0$, these conditions read

$$\gamma_i=\gamma_0\Gamma_i(\xi, \eta) \qquad (i=1, 2, \ldots, 21) \quad (27)$$

where $\Gamma_i(\xi, \eta)$ $(i=1, 2, \ldots, 21)$ are some arbitrary functions of ξ and η to be determined later. In the determination of α, β, ξ, η and w, there exist some freedoms without loss of generality:

Remark (i): if $\alpha(x, y, t)$ has the form $\alpha=\beta(x, y, t)\Omega(\xi, \eta)+\alpha_0(x, y, t)$, then we can take $\Omega\equiv 0$ (by substituting $w\to w(\xi, \eta)-\Omega(\xi, \eta)$).

Remark (ii): if $\beta(x, y, t)$ has the form $\beta=\beta_0(x, y, t)\Omega(\xi, \eta)$, then we can take $\Omega\equiv\Omega_0=$ constant (by substituting $w\to w(\xi, \eta)\Omega_0/\Omega(\xi, \eta)$).

Remark (iii): if $\xi=\xi(\xi_0(x, y, t), \eta)$ (or $\eta=\eta(\xi, \eta_0(x, y, t))$), then we can take $\xi=\xi_0$ (or $\eta=\eta_0$) (by taking $w(\xi(\xi_0, \eta), \eta)$ or $w(\xi, \eta(\xi, \eta_0))\to w(\xi_0, \eta)$ or $w(\xi, \eta_0)$).

Remark (iv): if $\xi(x, y, t)$ (or $\eta(x, y, t)$) is determined by an equation of the form $\Omega(\xi)=\xi_0(x, y, t)$ (or $\Omega(\eta)=\eta_0(x, y, t)$) where $\Omega(\xi)$ (or $\Omega(\eta)$) is any invertible function, then one can taken $\Omega(\xi)=\xi$ (or $\Omega(\eta)=\eta$) (by taking $\xi\to\Omega^{-1}(\xi)$ or $\eta\to\Omega^{-1}(\eta)$).

It is necessary to point out that each freedom will be fixed by using the corresponding remark once, that is to say, remarks (i) and (ii) can be used only once and remarks (iii) and (iv) can be used only twice (once for ξ and the other for η); more use of the remarks will result in loss of generality.

Using remarks (i)-(iv) to fix the freedoms in the determination of α, β, ξ, η and w and analysing (27) carefully, we get the only possible three types of non-equivalent solutions of (27) (see the appendix for details of the derivation):

(i)
$$u=-\frac{1}{6}\theta^4Z_1^2x^2+\left(\frac{\theta_t}{6\theta}-\frac{Z_1}{6}(\eta_t+2\theta^3Z_1\sigma+\theta^3Z_2)\right)x$$
$$+\frac{\sigma_t}{6\theta}-\frac{1}{6\theta^2}\left(\frac{1}{2\theta}\eta_t+\theta^2Z_1\sigma+\tfrac{1}{2}\theta^2Z_2\right)^2+\theta^2w(\xi, \eta) \quad (28)$$

$$\xi=\theta x+\sigma \qquad \theta_y=\theta^3Z_1 \qquad \eta=\int^y\theta^2(y', t)\,\mathrm{d}y'+\eta_0(t) \quad (29a, b, c)$$

$$\sigma=\exp\left(\int^y\theta^2(y', t)Z_1(\eta(y', t))\,\mathrm{d}y'\right)$$
$$\times\left[\sigma_0(t)+\int^y\frac{1}{2\theta(y_1, t)}[\eta_t(y_1, t)+\theta^3(y_1, t)Z_2(\eta(y_1, t))]\right.$$
$$\left.\times\exp\left(\int^{y_1}\theta^2(y', t)Z_1(\eta(y', t))\,\mathrm{d}y'\right)\mathrm{d}y_1\right] \quad (30)$$

and $w(\xi, \eta)$ is determined by

$$\begin{aligned}
&w_{\xi\xi\xi\xi}+6(ww_\xi)_\xi+w_{\eta\eta}+(2Z_1\xi+Z_2)w_{\xi\eta}+6Z_1w_\eta \\
&\quad+[(3Z_1^2+Z_{1\eta})\xi+\tfrac{3}{2}Z_1Z_2+\tfrac{1}{2}Z_{2\eta}]w_\xi+(6Z_1^2+2Z_{1\eta})w \\
&\quad=(3Z_1^4+4Z_1^2Z_{1\eta}+\tfrac{1}{3}Z_{1\eta}^2+\tfrac{1}{3}Z_1Z_{1\eta\eta})\xi^2+(\tfrac{1}{3}Z_{1\eta}Z_{2\eta}+\tfrac{1}{6}Z_2Z_{1\eta\eta}+\tfrac{1}{6}Z_1Z_{2\eta\eta} \\
&\quad+\tfrac{5}{2}Z_1Z_2Z_{1\eta}+3Z_1^3Z_2+\tfrac{3}{2}Z_1^2Z_{2\eta})\xi+\tfrac{1}{4}Z_{1\eta}Z_2^2 \\
&\quad+\tfrac{3}{4}Z_1^2Z_2^2+\tfrac{3}{4}Z_1Z_2Z_{2\eta}+\tfrac{1}{12}Z_{2\eta}^2+\tfrac{1}{12}Z_2Z_{2\eta\eta}
\end{aligned}\tag{31}$$

where $Z_1(\eta)$ and $Z_2(\eta)$ are arbitrary functions of η and $\eta_0(t)$ and $\sigma_0(t)$ are any functions of t.

(ii)
$$u=\frac{1}{6\theta^2}[\theta\theta_tx+(\sigma_{2t}\theta-4\sigma_2^2)y^2+(\theta\sigma_{1t}+4\sigma_2\sigma_1)y+(\theta\sigma_{0t}-\sigma_1^2)] + \theta^2w(\xi,\eta)\tag{32}$$

$$\xi=\theta x+\sigma_2y^2+\sigma_1y+\sigma_0\tag{33}$$

$$\eta=\int^t\theta^3(t')\,\mathrm{d}t'\tag{34}$$

$$\sigma_2=\tfrac{1}{2}(F\theta^4+\theta_t)\tag{35}$$

and the $w(\xi, \eta)$ equation is

$$[w_{\xi\xi\xi}+6ww_\xi-w_\eta]_\xi+F(\eta)w_\xi+\tfrac{1}{3}(\tfrac{1}{2}F_\eta-F^2)=0\tag{36}$$

with $F(\eta)$ being an arbitrary function of η, and θ, σ_1, σ_0 being any functions of t.

(iii)
$$u=\alpha_2(y,t)x^2+\alpha_1(y,t)x+w(y,t)\tag{37}$$

and $\alpha_2(y, t)$, $\alpha_1(y, t)$ and $w(y, t)$ satisfy

$$\alpha_{2yy}+36\alpha_2^2=0\tag{38}$$

$$\alpha_{1yy}+36\alpha_2\alpha_1-2\alpha_{2t}=0\tag{39}$$

$$w_{yy}+12\alpha_2w+6\alpha_1^2-\alpha_{1t}=0.\tag{40}$$

The reduction equations (31), (36) and (40) can be simplified further. In order to simplify these reduction equations, we cannot take directly some constraints on the arbitrary functions that appeared in these equations because we have fixed all the freedoms by using all the remarks (see the appendix), except remark (iv) for ξ, which cannot be used to fix these arbitrary functions. Nevertheless, we can simplify the reduction equations by some transformations of the dependent and independent variables, which is equivalent to the different uses of the remarks. For the sake of seeing this property more clearly, we discuss first the third type of similarity reductions of the KPE.

For the third type of similarity reduction of the KPE, the transformations

$$w\to C_1(t)B(y,t)+\int^yB^{-2}(y_1,t)\int^{y_1}[\alpha_{1t}(y',t)-6\alpha_1^2(y',t)]B^2(y',t)\,\mathrm{d}y'\,\mathrm{d}y_1 + B(y,t)q(Z(y,t),t)\tag{41a}$$

$$Z(y,t)=C_2(t)\int^yB^{-2}(y',t)\,\mathrm{d}y'+Z_0(t)\tag{41b}$$

where $B(y, t)$ is determined by

$$B_{yy} + 12\alpha_2 B = 0 \tag{41c}$$

would simplify the reduction equation (40) to

$$q_{zz} = 0 \tag{41d}$$

which possesses the same form as that of the classical Lie approach but with the different independent variable Z instead of y; morever, some more arbitrary functions of t, $C_1(t)$, $C_2(t)$, $Z_0(t)$ and another two, which would appear in the solution of (41c) have been introduced in the solutions of the KPE which cannot be obtained by the classical Lie approach. For completeness we write down all the solutions in this case.

(iiia). The first type of special solution of (38)-(40) read

$$\alpha_2 = 0 \tag{42}$$

$$\alpha_1 = A_1(t)y + A_0(t) \tag{43}$$

and

$$u(x, y, t) = (A, y + A_0)x - \tfrac{1}{2}A_1^2 y^4 + \tfrac{1}{6}(A_{1t} - 12A_1A_0)y^3 + (A_{0t} - 6A_0^2)y^2 + K(t)y + L(t) \tag{44}$$

where $A_1(t)$, $A_0(t)$, $K(t)$ and $L(t)$ are all arbitrary functions of t. It is worth pointing out that the third type of reduction ((4.11) in [2]) obtained by the classical Lie approach corresponds only to a special case of the solution (44) for $A_1 = 0$.

(iiib). The second type of special solution of (38)-(40) is

$$\alpha_2 = -\tfrac{1}{6}(y + y_0(t))^{-2} \tag{45}$$

$$\alpha_1 = B_0(t)(y + y_0(t))^{-2} + B_1(t)(y + y_0(t))^3 - \tfrac{1}{6}y_{0t}(y + y_0(t))^{-1} \tag{46}$$

and

$$w(y, t) = C_1(t)(y + y_0)^2 + C_2(t)(y + y_0)^{-1} - \tfrac{3}{2}B_0^2(y + y_0)^{-2} - \tfrac{1}{9}B_1^2(y + y_0)^8 - 3B_0B_1(y + y_0)^3 + \tfrac{1}{2}B_1 y_{0t}(y + y_0)^4 - \tfrac{1}{2}B_{0t} + \tfrac{1}{18}(y + y_0)^5 + \tfrac{1}{12}y_{0tt}(y + y_0) \tag{47}$$

where $y_0(t)$, $B_0(t)$, $B_1(t)$, $C_1(t)$ and $C_2(t)$ are all independent arbitrary functions of t.

(iiic). The general solutions of equations (38)-(40) can be expressed by

$$\alpha_2 = -\wp(\sqrt{6}(y + y_0(t)); 0, g_3(t)) \tag{48}$$

$$\alpha_1 = \alpha_2\left(B_0(t) + B_1(t)\int^y \alpha_2^{-2}(y_1, t)\,\mathrm{d}y_1 + \int^y \alpha_2^{-2}(y_2, t)\,\mathrm{d}y_2 \int^{y_2} [\alpha_2^2(y_1, t)]_t\,\mathrm{d}y_1\right) \tag{49}$$

and the w equation is given by the linear equation (40) with α_2 and α_1 given by (48) and (49), where $y_0(t)$, $g_3(t)$, $B_0(t)$, $B_1(t)$ and another two integral functions of w equation are arbitrary functions of t, and $\wp(\tau; g_2, g_3)$ is the Weierstrass elliptic function which is defined by

$$\left(\frac{\mathrm{d}\wp}{\mathrm{d}\tau}\right)^2 = 4\wp^3 - g_2\wp - g_3 \tag{50}$$

while $w(y, t)$ satisfies a Lamé equation (40) with α_2 and α_1 given by (48) and (49). The details of the solution of the Lamé equation can be seen from [6].

It is clear that the latter two subcases, (iiib) and (iiic), are all new solutions of the KPE which have not yet been obtained by the classical Lie approach.

Analogously to the third type of similarity reductions of the KPE, if we take the transformations

$$w \to \tfrac{1}{6}Z_1^2(\eta)\xi^2 + \tfrac{1}{6}Z_1(\eta)Z_2(\eta)\xi + \tfrac{1}{24}Z_2^2(\eta) + \theta_1^2(\eta)q(Z, T) \tag{51a}$$

$$\theta_1 = \exp\left(-\int^{\eta} Z_1(\eta')\,\mathrm{d}\eta'\right) \tag{51b}$$

$$Z = \theta_1(\eta)\xi - \tfrac{1}{2}\int^{\eta} Z_2(\eta_1)\,\mathrm{d}\eta_1 \exp\left(-2\int^{\eta_1} Z_1(\eta')\,\mathrm{d}\eta'\right) \tag{51c}$$

and

$$T = \int^{\eta} \mathrm{d}\eta_1 \exp\left(-2\int^{\eta_1} Z_1(\eta')\,\mathrm{d}\eta'\right) \tag{51d}$$

then the first type of reduction equation (31) would be simplified to the usual Boussinesq equation

$$q_{ZZZZ} + 6q_Z^2 + 6qq_{ZZ} + q_{TT} = 0 \tag{52}$$

which has the same form as the reduction equation obtained by the classical Lie approach but with different independent variables for $Z_1 \neq 0$ and $Z_2 \neq 0$. As with the third type of reduction solution, two arbitrary functions $Z_1(\eta)$ and $Z_2(\eta)$ cannot be absorbed by other arbitrary functions which have been introduced by the classical Lie approach. For completeness one can also substitute the exact solutions of q equation (52) (such as the trivial solution $q=0$ and the single soliton solution) into the final result (28) to see that the results are quite different for Z_i $(i=1, 2)$ being zero or not, but we do not do so here.

In the second type of reduction, the transformations

$$w \to \frac{F(\eta)}{6}\xi + \theta_1^2(\eta)q(Z, T) \tag{53a}$$

$$\theta_1 = \exp\int^{\eta} F(\eta')\,\mathrm{d}\eta' \tag{53b}$$

$$Z = \theta_1\xi \tag{53c}$$

and

$$T = \int^{\eta} \theta_1^3(\eta')\,\mathrm{d}\eta' \tag{53d}$$

would simplify the w equation (36) to the once-differentiated KdVE

$$q_{ZZZ} + 6qq_Z - q_T)_Z = 0 \tag{53e}$$

which has also the same form as that of the classical Lie approach but with different independent variables, except $F=0$. As discussed before the other two cases, the constraint $F=0$ (as in the classical Lie approach) will result in loss of generality and we do not discuss it further.

In summary, all three types of the similarity reductions of the KPE obtained by the classical Lie approach are only special cases of the results obtained by the direct method. Although their final reduction equations have the same form, some more arbitrary functions can be introduced into the solutions of the KPE in addition to those introduced by the classical Lie approach.

In [5] and [7], we have reported all the possible one-dimensional (i.e. ODEs) reductions of the KPE. For the sake of completeness we list the main results here.

(I)
$$u(x,y,t)=\theta^2(y,t)w(z)+\frac{1}{6\theta^2}[(\theta_t x+\sigma_t)\theta-(\theta_y x+\sigma_y)^2] \tag{54a}$$

$$Z=Z(x,y,t)=\theta(y,t)x+\sigma(y,t) \tag{54b}$$

$$\theta_{yy}=A\theta^5 \tag{54c}$$

$$-\theta_t+\sigma_{yy}=(A\sigma+B)\theta^4 \tag{54d}$$

and w satisfies the ODE

$$\frac{\mathrm{d}^{(4)}}{\mathrm{d}Z^{(4)}}w+6\left(\frac{\mathrm{d}}{\mathrm{d}Z}w\right)^2+6w\frac{\mathrm{d}^2}{\mathrm{d}Z^2}w+(Az+B)\frac{\mathrm{d}}{\mathrm{d}Z}w+2Aw=-(Az+B)^2 \tag{55}$$

where A and B are arbitrary constants. All the possible solutions of (54) and (55) have been given in [5].

(II)
$$u=\alpha(x,y,t)+\beta(x,y,t)w(Z(y,t)) \qquad (Z_y\neq 0) \tag{56}$$

$$-\beta_x Z_t+2\beta_y Z_y+\beta Z_{yy}=\beta Z_y^2\Gamma_a(z) \tag{57}$$

$$-\beta_{tx}+12\alpha_x\beta_x+6\alpha\beta_{xx}+6\beta\alpha_{xx}+\beta_{xxxx}+\beta_{yy}=\beta Z_y^2\Gamma_b(z) \tag{58}$$

$$6\beta_x^2+6\beta\beta_{xx}=\beta Z_y^2\Gamma_c(z) \tag{59}$$

$$-\alpha_{tx}+6\alpha_x^2+6\alpha\alpha_{xx}+\alpha_{xxxx}+\alpha_{yy}=\beta Z_y^2\Gamma_d(z) \tag{60}$$

and $w(z)$ satisfies the ODE

$$\frac{\mathrm{d}^2}{\mathrm{d}Z^2}w+\Gamma_a(z)\frac{\mathrm{d}w}{\mathrm{d}z}+\Gamma_b(z)w+\Gamma_c(z)w^2+\Gamma_d(z)=0 \tag{61}$$

where Γ_a, Γ_b, Γ_c and Γ_d are some functions of z.

(III)
$$u=\alpha(x,y,t)+\beta(x,y,t)w(t) \tag{62}$$

$$-\beta_{tx}+12\alpha_x\beta_x+6\alpha\beta_{xx}+6\beta\alpha_{xx}+\beta_{xxxx}+\beta_{yy}=-\beta_x\Gamma_A(t) \tag{63}$$

$$6\beta_x^2+6\beta\beta_{xx}=-\beta_x\Gamma_B(t) \tag{64}$$

$$-\alpha_{tx}+6\alpha_x^2+6\alpha\alpha_{xx}+\alpha_{xxxx}+\alpha_{yy}=-\beta_x\Gamma_C(t) \tag{65}$$

where Γ_A, Γ_B and Γ_C are some functions of t and $w(t)$ satisfies the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}w+\Gamma_A(t)w-\Gamma_B(t)w^2+\Gamma_C(t)=0$$

for $\beta_x\neq 0$ and $w(t)$ is an arbitrary function of t for $\beta_x=0$. All the possible solutions of the (57)-(65) have been given in [7].

3. Summary and discussion

In this paper, we have obtained all the possibilities to reduce a (2+1)-dimensional equation, the KPE, to some PDEs in two variables. There are three types of similarity reductions of the KPE which are parallel to the classical Lie approach. In the first type of reduction, the reduction equation is a variable coefficient Boussinesq equation which is equivalent to the first type of similarity reduction, the Boussinesq equation, obtained by the classical Lie approach but with different independent variables; two more arbitrary functions are included in the solutions of the KPE. The solutions are same as the results obtained by the classical Lie approach only for these two arbitrary functions vanishing. The second type of similarity reduction equation is a variable coefficient once-differentiated KdVE which can be transformed to the constant coefficient once-differentiated KdVE by some transformations of dependent and independent variables. One more arbitrary function of t is introduced into the solutions of the KPE for the second type of reduction obtained by the classical Lie approach. The third type of similarity reduction equation is only a second-order PDE which includes three subcases. The third type of similarity reduction of the KPE obtained by the classical Lie approach is only a special case of the first subcase for the one arbitrary function of t taken as zero.

In summary, in our results, additional arbitrary functions can be included in the similarity reductions which are missed by the classical Lie approach. How to get all the similarity reductions by the classical Lie approach [2] or the non-classical symmetry reduction method [8, 9], rather than the direct method, should be studied further.

The one-dimensional similarity reductions (54)–(65) of the KPE can be obtained directly by assuming w in (3) is η independent or making $\xi=\eta$ in (4) with (5)–(26). These results can also be obtained by using the direct method once again to reduce the (1+1)-dimensional PDEs obtained here to ODEs.

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Appendix. The derivations of three types of reductions

Combining (27) with $i=2$ and remark (ii), one can get immediately

$$\beta=\xi_x^2 \qquad \Gamma_2=1. \tag{A1}$$

Substituting (A1) into (27) with $i=5$, we have

$$\eta_x=\xi_x\Gamma_5(\xi,\eta) \tag{A2}$$

Integrating (A2) with respect to x leads to

$$\begin{aligned}\eta&=\int^x \xi_{x'}(x',y,t)\Gamma_5(\xi(x',y,t),\eta(x',y,t))\,\mathrm{d}x'+\eta_0^1(y,t)\\ &=\eta_0^1(y,t)+\Gamma_5(\xi,\eta)(\xi+\eta_1(y,t))-\int^x \xi(x',y,t)\\ &\quad\times[\Gamma_{5\xi}(\xi(x',y,t),\eta(x',y,t))\xi_{x'}(x',y,t)+\Gamma_{5\eta}(\xi(x',y,t),\eta(x',y,t))\\ &\quad\times\eta_{x'}(x',y,t)]\,\mathrm{d}x'\equiv\Gamma_5(\xi,\eta)\xi+\eta_0(y,t).\end{aligned} \tag{A3}$$

Remark (iii) for η tells us that if $\eta=\Omega(\xi,\eta(\xi,\eta_0))$, one can take $\eta\equiv\eta_0$, this means that from (A3) one can take

$$\Gamma_5(\xi,\eta)\equiv 0 \qquad \eta=\eta_0=\eta_0^1(y,t). \tag{A4}$$

Due to (A1), (27) with $i=9$ becomes

$$(\ln \xi_x)_x = \tfrac{1}{9}\Gamma_9(\xi, \eta)\xi_x. \tag{A5}$$

As η is x independent, integrating (A5) with respect to x yields

$$\begin{aligned}\Omega(\xi, \eta) &\equiv \int^{\xi} \exp\left(-\tfrac{1}{9}\int^{\xi_1} \Gamma_9(\xi', \eta)\,\mathrm{d}\xi'\right)\mathrm{d}\xi_1 \\ &= \theta(y, t)x + \sigma(y, t) \equiv \xi_0(x, y, t).\end{aligned} \tag{A6}$$

Remark (iii) for ξ tells us that we can take

$$\Gamma_9(\xi, \eta) \equiv 0 \qquad \xi = \xi_0 = \theta(y, t)x + \sigma(y, t). \tag{A7}$$

Substituting (A1), (A4) and (A7) into (27) with $i = 19$ and using the remark (i) yields

$$\alpha = \frac{1}{6\theta^2}[\theta(\theta_t x + \sigma_t) - (\theta_y x + \sigma_y)^2] \qquad \Gamma_{19} = 0. \tag{A8}$$

According to the results (A1), (A4), (A7) and (A8), we have

$$\Gamma_1 = \Gamma_3 = \Gamma_4 = \Gamma_6 = \Gamma_7 = \Gamma_8 = \Gamma_{10} = \Gamma_{11} = \Gamma_{12} = \Gamma_{13} = \Gamma_{14} = 0 \tag{A9}$$

as well. The remaining equations in (27) read

$$-\alpha_{tx} + 6\alpha_x^2 + 6\alpha\alpha_{xx} + \alpha_{yy} + \alpha_{xxxx} = \theta^6\Gamma_{15}(\xi, \eta) \tag{A10}$$

$$6\theta^2\alpha_{xx} + 2\theta_y^2 + 2\theta\theta_{yy} = \theta^6\Gamma_{16}(\xi, \eta) \tag{A11}$$

$$-3\theta\theta_t + 12\theta^2\alpha_x + 4\theta_y(\theta_y x + \sigma_y) + \theta(\theta_{yy}x + \sigma_{yy}) = \theta^5\Gamma_{17}(\xi, \eta) \tag{A12}$$

$$4\theta_y\eta_y + \theta\eta_{yy} = \theta^5\Gamma_{18}(\xi, \eta) \tag{A13}$$

$$\eta_y^2 = \theta^4\Gamma_{20}(\xi, \eta) \tag{A14}$$

and

$$-\eta_t\theta + 2(\theta_y x + \sigma_y)\eta_y = \theta^4\Gamma_{21}(\xi, \eta). \tag{A15}$$

And all the freedoms have been fixed except the freedoms in remark (iv).

To solve the remaining equations (A10)–(A15), there are two possibilities to discuss further:

(A) $\Gamma_{20} \neq 0$. In this case, (A14) can be rewritten as

$$\pm\eta_y[\Gamma_{20}(\eta)]^{-1/2} = \theta^2(y, t). \tag{A16}$$

Integrating (A16) with respect to y once and using remark (iv) for η, we get

$$\Gamma_{20} = 1 \tag{A17}$$

$$\eta_y = \theta^2 \quad \text{or} \quad \eta = \int^{y} \theta^2(y', t)\,\mathrm{d}y' + \eta_0(t) \tag{A18}$$

with $\eta_0(t)$ being any function of t.

Substituting (A18) into (A13) yields

$$\theta_y = \tfrac{1}{6}\Gamma_{18}(\xi, \eta)\theta^3 \equiv Z_1(\eta)\theta^3 \tag{A19}$$

i.e. $\Gamma_{18} = 6Z_1$ is an arbitrary function of η and independent of ξ. Combining the results of (A8) and (A19), the solution of (A11) reads

$$\Gamma_{16}(\xi, \eta) = \Gamma_{16}(\eta) = 6Z_1^2 + 2Z_{1\eta}. \tag{A20}$$

Since the functions η and θ are x independent and the LHSs of (A12) and (A15) are x dependent only in the linear forms, the only possible forms of Γ_{21} and Γ_{17} are

$$\Gamma_{21}(\xi, \eta) = Z_3(\eta)\xi + Z_2(\eta) \tag{A21}$$

and

$$\Gamma_{17}(\xi, \eta) = Z_4(\eta)\xi + Z_5(\eta). \tag{A22}$$

It is quite straightforward to determine the functions $Z_i(\eta)$ $(i = 2, 3, 4, 5)$, and the results read

$$Z_3 = 2Z_1 \qquad Z_4 = 3Z_1^2 + Z_{1\eta} \qquad Z_5 = \tfrac{3}{2}Z_1Z_2 + \tfrac{1}{2}Z_{2\eta} \tag{A23}$$

and Z_2 remains free while η, σ and θ satisfy the condition

$$\sigma_y = \frac{1}{2\theta}[\eta_t + \theta^3(2Z_1\sigma + Z_2)] \tag{A24}$$

or

$$\sigma = \exp\left(\int^y \theta^2(y', t)Z_1(\eta(y', t))\,\mathrm{d}y'\right)\left[\sigma_0(t) + \int \frac{\mathrm{d}y_1}{2\theta(y_1, t)}[\eta_t(y_1, t) + \theta^3 Z_2(\eta(y_1, t))] \times \exp\left(-\int^{y_1} \theta^2(y', t)Z_1(\eta(y', t))\,\mathrm{d}y'\right)\right] \tag{A25}$$

with $\sigma_0(t)$ being an arbitrary function of t.

Similarly, the only possibility for Γ_{15} in (A10) is

$$\Gamma_{15}(\xi, \eta) = Z_6(\eta)\xi^2 + Z_7(\eta)\xi + Z_8(\eta). \tag{A26}$$

Substituting (A8), (A18), (A19), (A25) and (A26) into (A10) and comparing the coefficients of x^2, x and x^0, we have

$$\alpha_{2yy} + 36\alpha_2^2 = \theta^8 Z_6 \tag{A27}$$

$$\alpha_{1yy} + 36\alpha_2\alpha_1 = 2\alpha_{2t} + \theta^7(2Z_6\sigma + Z_7) \tag{A28}$$

and

$$\alpha_{0yy} + 12\alpha_2\alpha_0 + 6\alpha_1^2 - \alpha_{1t} = \theta^6(Z_6\sigma^2 + Z_7\sigma + Z_8) \tag{A29}$$

where α_2, α_1 and α_0 are defined by

$$\alpha = \alpha_2 x^2 + \alpha_1 x + \alpha_0 \tag{A30}$$

i.e.

$$\alpha_2 \equiv -\tfrac{1}{6}\theta^4 Z_1^2 \tag{A31}$$

$$\alpha_1 \equiv -\tfrac{1}{6}Z_1(\eta_t + 2\theta^3 Z_1\sigma + \theta^3 Z_2) + \frac{\theta t}{6\theta} \tag{A32}$$

and

$$\alpha_0 \equiv \frac{\sigma_t}{6\theta} - \frac{1}{6\theta^2}\left(\frac{1}{2\theta}\eta_t + \theta^2 Z_1\sigma + \tfrac{1}{2}\theta^2 Z_2\right)^2. \tag{A33}$$

After some simple but tedious calculations one can get the solutions of (A27)–(A29):

$$Z_6 = -3Z_1^4 - 4Z_1^2 Z_{1\eta} - \tfrac{1}{3}Z_{1\eta}^2 - \tfrac{1}{3}Z_1 Z_{1\eta\eta} \tag{A34}$$

$$Z_7 = -\tfrac{1}{3}Z_{1\eta}Z_{2\eta} - \tfrac{1}{6}Z_2 Z_{1\eta\eta} - \tfrac{1}{6}Z_1 Z_{2\eta\eta} - \tfrac{5}{2}Z_1 Z_2 Z_{1\eta} - 3Z_1^3 Z_2 - \tfrac{3}{2}Z_1^2 Z_{2\eta} \tag{A35}$$

and

$$Z_8 = -\tfrac{1}{4}Z_{1\eta}Z_2^2 - \tfrac{3}{4}Z_1^2Z_2^2 - \tfrac{3}{4}Z_1Z_2Z_{2\eta} - \tfrac{1}{12}Z_{2\eta}^2 - \tfrac{1}{12}Z_2Z_{2\eta\eta}. \quad \text{(A36)}$$

Now collecting all the results we have obtained, the first type of similarity reductions given in section 2 follows immediately. From the derivation of this type of solution, we see that we have not used remark (iv) for ξ; however, this remark cannot make us fix any functions Z_1 and Z_2.

(B) $\Gamma_{20}(\xi, \eta) = 0$. In this case, (A14) and (A13) lead to

$$\eta_y = 0 \qquad \Gamma_{18} = 0 \quad \text{(A37)}$$

i.e.

$$\eta = \eta(t) \quad \text{or} \quad t = f(\eta) \quad \text{(A38)}$$

with f being the inverse function of $\eta(t)$. Because of (A38) and (A15) we have

$$\theta_y = 0 \quad \text{(A39)}$$

$$-\frac{d\eta}{\Gamma_{21}(\eta)} = \theta^3 \, dt. \quad \text{(A40)}$$

Combining the integration of (A40) and remark (iv) for η we have

$$\eta = \int^t \theta^3(t') \, dt' \qquad \Gamma_{21} = -1. \quad \text{(A41)}$$

In consequence of (A39), (A8) becomes

$$\alpha = \frac{1}{6\theta^2}[\theta(\theta_t x + \sigma_t) + \sigma y^2]. \quad \text{(A42)}$$

Substituting (A39) and (A42) into (A11) and (A12) yields

$$\Gamma_{16} = 0 \quad \text{(A43)}$$

$$\sigma = \sigma_2(t)y^2 + \sigma_1(t)y + \sigma_0(t) \quad \text{(A44)}$$

with

$$\Gamma_{17}(\eta) \equiv F(\eta) \qquad \sigma_2(t) = \tfrac{1}{2}[\theta^4 F(\eta(t)) + \theta_t] \quad \text{(A45)}$$

and $\sigma_1(t)$ and $\sigma_0(t)$ being some arbitrary functions of t. While (A10) yields the result

$$\Gamma_{15} = \tfrac{1}{3}(\tfrac{1}{2}F_\eta - F^2). \quad \text{(A46)}$$

Collecting the results in this case, we get the second type of similarity reductions of the KPE shown by (32)–(36). In this case, we have also used remarks (i)–(iii) and (iv) for η. The remaining remark (iv) for ξ cannot be used to fix $F(\eta)$, the arbitrary function of η.

(C) In order to obtain the third type of reduction we have to discuss the $\xi_x = 0$ case. In this case, we can suppose that $\eta_x = 0$ at the same time, otherwise exchange of ξ and η will lead to the first and the second types of reductions again. Furthermore, we suppose that $\xi_y \neq 0$ (or $\eta_y \neq 0$), otherwise w would be a function of t only, this case has been given in (62)–(65).

The conditions $\xi_x = 0$ and $\eta_x = 0$ simplify (4) with (5)–(26) to

$$\beta\xi_y^2 w_{\xi\xi} + \beta\eta_y^2 w_{\eta\eta} + 2\beta\xi_y\eta_y w_{\xi\eta} + (-\beta_x\eta_t + 2\beta_y\eta_y + \beta\eta_{yy})w_\eta$$
$$+(-\beta_x\xi_t + 2\beta_y\xi_y + \beta\xi_{yy})w_\xi$$
$$+(-\beta_{tx} + 12\alpha_x\beta_x + 6\alpha\beta_{xx} + 6\alpha_{xx}\beta + \beta_{yy} + \beta_{xxxx})w$$
$$+6(\beta_x^2 + \beta\beta_{xx})w^2 + (-\alpha_{tx} + 6\alpha_x^2 + 6\alpha\alpha_{xx} + \alpha_{yy} + \alpha_{xxxx}) = 0. \quad \text{(A47)}$$

Since $\xi_y \neq 0$, (A47), being a PDE can be written as

$$\eta_y^2 = \xi_y^2\Gamma_A(\xi, \eta) \quad \text{(A48)}$$

$$\eta_y = \xi_y\Gamma_B(\xi, \eta) \quad \text{(A49)}$$

$$\beta_x^2 + \beta\beta_{xx} = \beta\xi_y^2\Gamma_C(\xi, \eta) \quad \text{(A50)}$$

$$-\beta_x\xi_t + 2\beta_y\xi_y + \beta\xi_{yy} = \beta\xi_y^2\Gamma_D(\xi, \eta) \quad \text{(A51)}$$

$$-\beta_x\eta_t + 2\beta_y\eta_y + \beta\eta_{yy} = \beta\xi_y^2\Gamma_E(\xi, \eta) \quad \text{(A52)}$$

$$-\beta_{tx} + 12\alpha_x\beta_x + 6\alpha_{xx}\beta + 6\alpha\beta_{xx} + \beta_{yy} + \beta_{xxxx} = \beta\xi_y^2\Gamma_F(\xi, \eta) \quad \text{(A53)}$$

and

$$-\alpha_{tx} + 6\alpha_x^2 + 6\alpha\alpha_{xx} + \alpha_{yy} + \alpha_{xxxx} = \beta\xi_y^2\Gamma_G(\xi, \eta). \quad \text{(A54)}$$

In this case, let us consider remark (iii) at first from $\xi_x = 0$, i.e. $\xi = \xi(y, t)$, we can solve y explicitly,

$$y = y(\xi, t) \quad \text{(A55)}$$

hence

$$\eta = \eta(y, t) = \eta(y(\xi, t), t) \equiv \eta_1(\xi, t) \equiv \eta_1(\xi, \eta_0). \quad \text{(A56)}$$

According to remark (iii) for η we have

$$\eta = \eta_0 = t. \quad \text{(A57)}$$

And then

$$\xi = \xi(y, t) \equiv \xi(\xi_0, \eta). \quad \text{(A58)}$$

Using remark (iii) for ξ we get

$$\xi = y. \quad \text{(A59)}$$

Substituting (A57) and (A59) into (A52) and integrating it once with respect to y yields

$$\beta = \beta_0(y, t)\exp(-\Gamma_E(y, t)x). \quad \text{(A60)}$$

Combining (A60) and (A50) leads to

$$2\Gamma_E^2(y, t)\beta_0^2(y, t)\exp(-2\Gamma_E(y, t)x) = \beta_0(y, t)\exp(-\Gamma_E(y, t)x)\Gamma_C(y, t). \quad \text{(A61)}$$

Equation (A61) is true for any x only for

$$\Gamma_E(y, t) = \Gamma_C(y, t) = 0. \quad \text{(A62)}$$

Now using remark (ii) for (A60) with $\Gamma_E = 0$, we can take

$$\beta = \beta_0(y, t) = 1. \quad \text{(A63)}$$

Collecting the results obtained in this case and substituting them into the remaining equations of (A48)–(A54), the third type of similarity reduction of the KPE shown in (37)–(40) follows immediately after using remark (i) once.

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